

Econ 802

Lecture Notes on Chapter 8

Greg Dow

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Here I will only deal with material from sections 8.1, 8.2, 8.3, 8.4 and 8.6 of Varian. It is not necessary to read any other sections of Chapter 8.

However, at one point I will skip ahead to section 9.5 in Chapter 9 in order to do a proof in the proper sequence (I'll let you know when I do this).

Section 8.1 should be straightforward enough that I don't need to discuss it directly. So I will start with section 8.2.

The Slutsky equation is a central result in consumer theory. It relates the slopes of the Marshallian demand functions (which depend upon prices and income) to the slopes of the Hicksian demand functions (which depend upon prices and utility).

Notice that at least in principle the Marshallian functions are observable (if we had the right data we could estimate them) while the Hicksian functions are not observable (we will not usually have data on utility levels).

(2)

The usual form of the Slutsky equation is:

$$\frac{\partial x_j(p, m)}{\partial p_i} = \underbrace{\frac{\partial h_j[p, v(p, m)]}{\partial p_i}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_j(p, m)}{\partial m} \cdot x_i(p, m)}_{\text{income effect}}$$

for any pair of goods i and j .

A few comments on this.

- ① we could have $i = j$ so the derivative on the left side is the slope of the Marshallian demand for good i with respect to its own price. However we are not limited to this; we could also have $i \neq j$.
- ② We frequently split the overall effect $\frac{\partial x_j(p, m)}{\partial p_i}$ into two pieces: a substitution effect involving the slope of a Hicksian demand and an income effect involving a change in m . The minus sign is included in the income effect.
- ③ In the income effect we multiply by the Marshallian demand $x_i(p, m)$. Notice that the subscript i here refers to the good whose price changed (we are differentiating with respect to p_i).
- ④ From the definition of Marshallian demands the derivative on the left side depends on (p, m) . But Hicksian demands depend on (p, u) . We obtain the correct utility level by using $u = v(p, m)$ from the indirect utility function.

Proof of the Slutsky Equation

Choose arbitrary prices $p^* > 0$ and income $m^* > 0$.
 Let x^* maximize utility at (p^*, m^*)
 and write $u^* = u(x^*) = v(p^*, m^*)$.

We know that for any prices p , it is true that

$$h_j(p, u^*) = x_j[p, e(p, u^*)]$$

[see identity (4) from p. 18 of my notes on Chapter 7].

This identity just says that the Hicksian demand for good j at utility u^* is always equal to the Marshallian demand for the same good at the income $e(p, u^*)$. Recall that this income is the minimum income needed to achieve utility u^* .

Because the above equation is an identity that holds for all price vectors p , we can differentiate it on both sides with respect to p_i . This gives

$$\frac{\partial h_j(p, u^*)}{\partial p_i} = \frac{\partial x_j[p, e(p, u^*)]}{\partial p_i} + \frac{\partial x_j[p, e(p, u^*)]}{\partial m} \frac{\partial e(p, u^*)}{\partial p_i}$$

Evaluate everything at p^* and replace $e(p^*, u^*)$ by m^* ; note that we have

$e(p^*, u^*) = e[p^*, v(p^*, m^*)] = m^*$ from identity (1) on p. 17 of my notes on Chapter 7.

(4)

This gives

$$\frac{\partial h_j(p^*, u^*)}{\partial p_i} = \frac{\partial x_j(p^*, m^*)}{\partial p_i} + \frac{\partial x_j(p^*, m^*)}{\partial m} \frac{\partial e(p^*, u^*)}{\partial p_i}$$

Now focus on the last derivative on the right side.

From Shephard's Lemma we have

$$\frac{\partial e(p^*, u^*)}{\partial p_i} = h_i(p^*, u^*)$$

[see p. 15 of my lecture notes on Chapter 7]

Furthermore the Hicksian demand at (p^*, u^*) is equal to the Marshallian demand at (p^*, m^*)

[see identity (3) on p. 18 of my notes on Chapter 7]

$$\begin{aligned} \text{so } \frac{\partial e(p^*, u^*)}{\partial p_i} &= h_i(p^*, u^*) = x_i[p^*, e(p^*, u^*)] \\ &= x_i(p^*, m^*) \end{aligned}$$

Make this substitution and rearrange the equation at the top of this page to get

$$\frac{\partial x_j(p^*, m^*)}{\partial p_i} = \frac{\partial h_j(p^*, u^*)}{\partial p_i} - \frac{\partial x_j(p^*, m^*)}{\partial m} x_i(p^*, m^*)$$

We can replace $u^* = v(p^*, m^*)$ which eliminates the utility in the derivative of the Hicksian demand. Also recall that (p^*, m^*) were chosen arbitrarily, so this is true for all (p, m) . Hence we can drop the stars.

(5)

This completes the proof of the Slutsky equation.

For a discrete price change Δp_i , people often use this equation to approximate the resulting discrete quantity change Δx_j :

$$\Delta x_j \approx \frac{\partial x_j(p, m)}{\partial p_i} \Delta p_i = \frac{\partial h_j(p, v(p, m))}{\partial p_i} \Delta p_i - \frac{\partial x_j(p, m)}{\partial m} x_i(p, m) \Delta p_i$$

The interpretation is that the impact of Δp_i on x_j can be decomposed into two parts:

① substitution effect: what happens to x_j when we hold utility constant and move along an indifference curve

② income effect: what happens to x_j when income changes and we jump from one indifference curve to another, holding the prices constant.

Note: I will show in a moment that when we have $i = j$, we have $\frac{\partial h_i}{\partial p_i} \leq 0$ so the

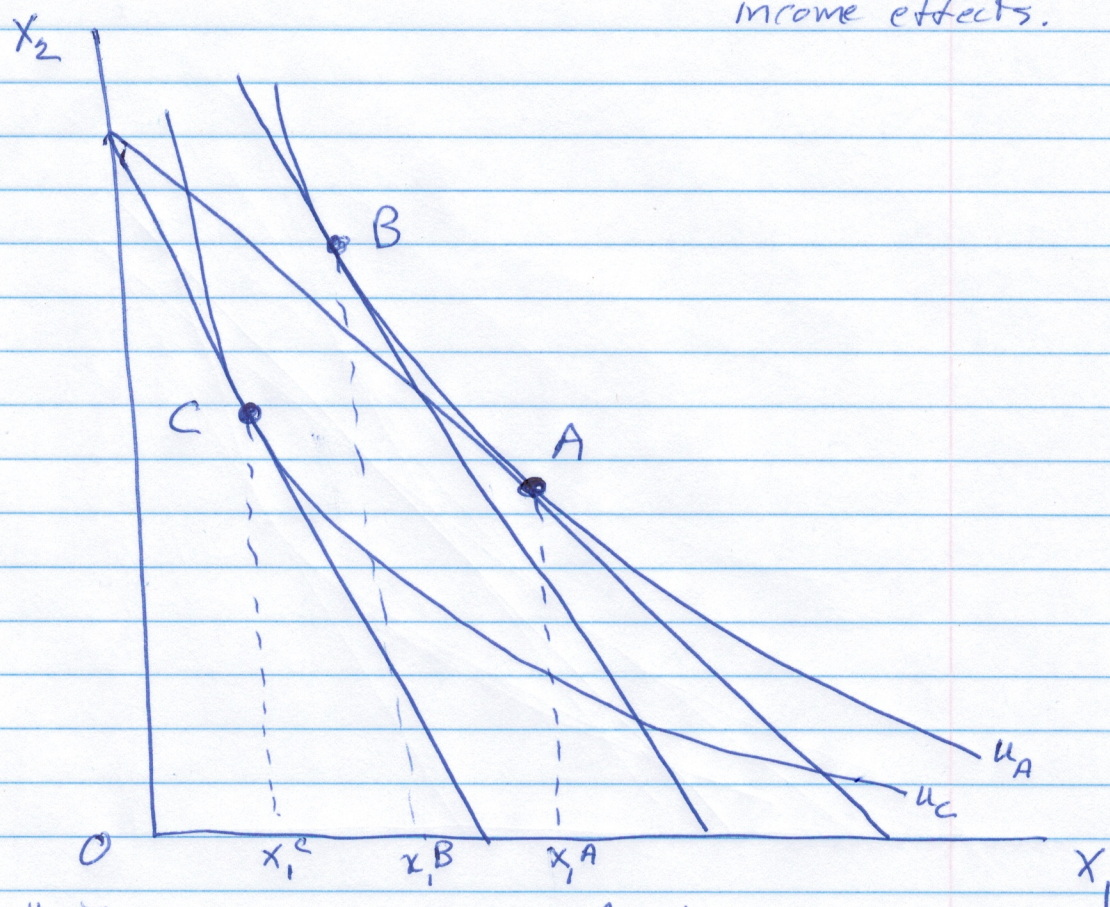
Hicksian demand curves slope down (or at least, not up). This implies that in this case the substitution effect will be non-positive.

(6)

Let's think about a two-good case graphically. Suppose $i = j = 1$. We will consider an increase in the price of good 1 so $\Delta p_1 > 0$. We want to know the overall effect on the quantity of good 1, which is Δx_1 . Approximately, the discrete change is

$$\Delta x_1 \approx \frac{\partial x_1}{\partial p_1} \Delta p_1$$

We would like to break this down into substitution and income effects.



Initially the consumer is at point A. When p_1 rises, the new budget line has the same vertical intercept but is steeper, and the optimal bundle is C. So the total effect on x_1 is $\Delta x_1 = x_1^C - x_1^A < 0$ (measured along the horizontal axis).

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To split this overall effect into substitution and income effects go back to the original bundle A on the indifference curve u_A . Construct a budget line that has the same slope as the one through point C (determined by the new price) but is tangent to the old indifference curve u_A . The tangency point occurs at B.

The substitution effect is $\Delta x_1^S = x_1^B - x_1^A < 0$.

Notice that this is obtained by moving along the old indifference curve u_A so utility does not change.

The income effect is $\Delta x_1^I = x_1^C - x_1^B < 0$.

This is obtained through a parallel shift in the budget line from B to C using the new prices (so it is like taking income away from the consumer while holding prices fixed).

I have drawn the graph so that x_1 is a normal good. (See section 8.1 in Varian). This means that with prices held constant the consumer wants more of x_1 when income rises, and less when income falls.

Under this assumption, the substitution and income effects go in the same direction as shown in the graph: both tend to reduce x_1 when p_1 increases.

However, if x_1 was an inferior good, a reduction in income would cause the consumer to increase demand for it.

In that case, point C would be to the right of B rather than to the left of it, although we would still have $\Delta x_1 < 0$ as long as the substitution effect is larger than the income effect.

It is Theoretically possible if x_1 is an inferior good, that the income effect could be so large that it exceeds the substitution effect. In this case the overall effect Δx_1 would be positive; when p_1 rises, so does consumption of good 1! This (very rare) situation is called a Giffen good (again see section 8.1).

Giffen goods are almost never observed (some economic historians claim that potatoes were a Giffen good during a famine in Ireland in the 1840s). But unfortunately, we cannot rule out an upward sloping Marshallian demand curve on purely theoretical grounds.

At this point you might be wondering whether consumer theory makes any predictions about observable behavior. The answer is yes, but it takes a little math to get there.

First write the Slutsky equation in matrix form:

$$\frac{\partial x(p, m)}{\partial p} = \frac{\partial h[p, v(p, m)]}{\partial p} - \frac{\partial x(p, m)}{\partial m} \cdot x(p, m)$$

Next, recall that from Shephard's Lemma we have

$$h(p, u) = \frac{\partial e(p, u)}{\partial p} \quad \text{Therefore} \quad \frac{\partial h(p, u)}{\partial p} = \frac{\partial^2 e(p, u)}{\partial p^2}$$

$$\text{or} \quad \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \dots & \frac{\partial h_1}{\partial p_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial p_1} & \dots & \frac{\partial h_n}{\partial p_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 e}{\partial p_1^2} & \dots & \frac{\partial^2 e}{\partial p_1 \partial p_n} \\ \vdots & & \vdots \\ \frac{\partial^2 e}{\partial p_n \partial p_1} & \dots & \frac{\partial^2 e}{\partial p_n^2} \end{bmatrix}$$

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Now use the properties of the expenditure function $e(p, u)$. We know that $e(p, u)$ is concave in prices, so its Hessian matrix $\frac{\partial^2 e(p, u)}{\partial p^2}$ is both symmetric and negative semi-definite.

Therefore $\frac{\partial h(p, u)}{\partial p}$ is both symmetric and neg semi-def.

Rearranging the Slutsky equation gives

$$\frac{\partial h(p, v(p, m))}{\partial p} = \frac{\partial x(p, m)}{\partial p} + \frac{\partial x(p, m)}{\partial m} \cdot x(p, m)$$

Because the left side is symmetric and neg semi-def, so is the right side. But the right side expression consists entirely of observable things: prices, income, and quantities demanded. In principle, if you had the data, you could estimate the Marshallian demand functions $x(p, m)$ take the derivatives and test the theoretical predictions about symmetry and negative semi-definiteness.

We can also say something else: because $\frac{\partial h(p, u)}{\partial p}$ is neg semi-def it has non-positive diagonal elements. This implies that Hicksian demand curves can never slope up. It also implies that when we are looking at the effect of an increase in p_i on x_i , the substitution effect can never be positive (it will usually be negative).

Because the substitution effect is non-positive, the only way to have a Giffen good is if (1) the good is inferior and (2) the income effect is larger in absolute value than the substitution effect.

Obtaining The direct utility function from The indirect utility function.

This is The last topic I will cover in Chapter 8. Sometimes you may not know The direct utility function $u(x)$, but you do know The indirect utility function $v(p, m)$. The question is: can we somehow compute $u(x)$ from our knowledge of $v(p, m)$? I will give you a method for doing this.

First, though, I need to prove a preliminary result about inverse demand functions. Varian does not cover this topic until section 9.5, so I am skipping down to that. You may want to read that section now so you will see what I am doing.

Suppose we are given some arbitrary interior consumption bundle $x^* > 0$. We want to find a price vector $p > 0$ and an income level $m > 0$ such that The consumer would choose x^* . I will assume The direct utility function is differentiable, quasi-concave and gives interior solutions when it is maximized (so we can ignore non-negativity constraints).

We know that if x^* is optimal for (p, m) then it is also optimal for (tp, tm) given any $t > 0$. This is true because The budget does not change. Since all that matters is The price vector relative to income we can simplify The problem by setting $m = 1$ and trying to find a price vector p such that x^* would be optimal.

Thus we want x^* to solve $\max_x u(x)$ subject to $px = 1$.

The FOCs are $\frac{\partial u(x^*)}{\partial x_i} = dp_i$ for all $i=1 \dots n$
and $px^* = 1$.

If these FOCs are satisfied then x^* is optimal at p and $m=1$ because we have quasi-concavity of $u(x)$ so there is no need to check the SOC; anything that satisfies FOC will be a solution.

From FOC, $\frac{\partial u(x^*)}{\partial x_i} x_i^* = dp_i x_i^*$ for all $i=1 \dots n$

Sum over i and use $\sum_i p_i x_i^* = 1$ to obtain

$$\sum_i \frac{\partial u(x^*)}{\partial x_i} x_i^* = d.$$

Substitute this back into FOC and solve for p_i :

$$p_i^* = \frac{\frac{\partial u(x^*)}{\partial x_i}}{d} = \frac{\frac{\partial u(x^*)}{\partial x_i}}{\sum_j \frac{\partial u(x^*)}{\partial x_j} x_j^*} \quad i=1 \dots n$$

This is called an inverse demand function.

It gives a vector of prices p^* such that the consumer would want to choose the given bundle x^* .

Notice that $p_i^* > 0$ for all i because the marginal utilities $\frac{\partial u(x^*)}{\partial x_i}$ are all positive.

Such an inverse demand function always exists (with $m=1$) as long as the direct utility function satisfies the assumptions described above.

Now let's return to the original question: if we know $v(p, m)$, can we compute $u(x)$?

I will prove that $u(x) = \min_p v(p, 1)$ subject to $p \cdot x = 1$.

Notice that we set $m=1$ in the indirect utility function, we minimize with respect to prices and we treat the consumption bundle x as a vector of constants in the constraint equation. Once you find the "optimal" prices, plug them back into $v(p, m)$ and this will give $u(x)$.

Here is a proof.

Choose any $x^* > 0$ and let p^* be a price vector at which x^* is optimal when $m=1$.

(This is why I had to discuss inverse demand functions first. If we didn't know about inverse demand, we would have no way of knowing that such a price vector p^* exists.)

Because x^* is optimal at prices p^* and income $m=1$, we have $u(x^*) = v(p^*, 1)$.

Let p be any other price vector satisfying the budget constraint when x^* is chosen; that is, any p such that $p \cdot x^* = 1$.

Then at the prices p , the bundle x^* is feasible, but not necessarily optimal.

This implies that it is always possible to find some feasible x' such that $u(x') \geq u(x^*)$

$$\text{Hence } v(p, 1) \geq v(p^*, 1) = u(x^*)$$

(with income $m=1$ and prices p , the consumer can always do at least as well as $u(x^*)$ and may be able to do better).

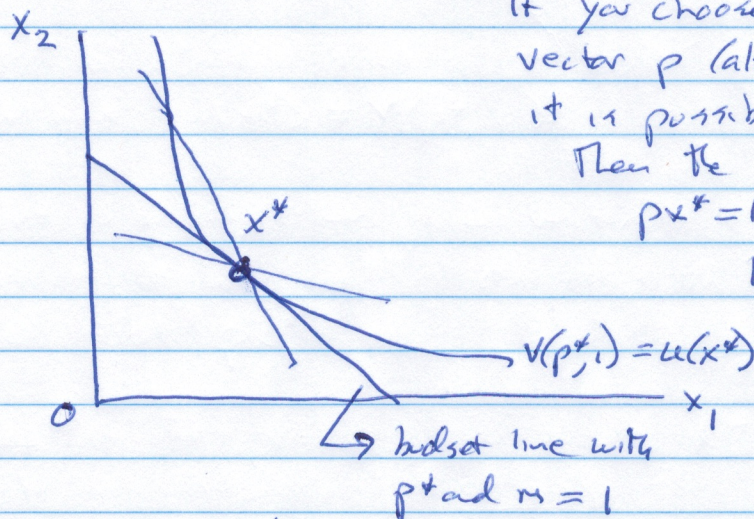
But then $v(p^*, 1) = u(x^*)$ is the minimum possible value of $v(p, 1)$ when $px^* = 1$

$$\text{So } u(x^*) = \min_p v(p, 1) \text{ subject to } px^* = 1$$

Since $x^* > 0$ was arbitrary, this holds for all $x > 0$, and

$$u(x) = \min_p v(p, 1) \text{ subject to } px = 1.$$

Here is some graphical intuition. Start from the bundle x^* and suppose it is optimal given p^* and $m=1$.



If you choose any other price vector p (also with $m=1$), and it is possible to buy x^* at p , then the budget line from $px^*=1$ must pass through x^* .

However the fact that we originally had a tangency point, so we are changing the slope of the budget

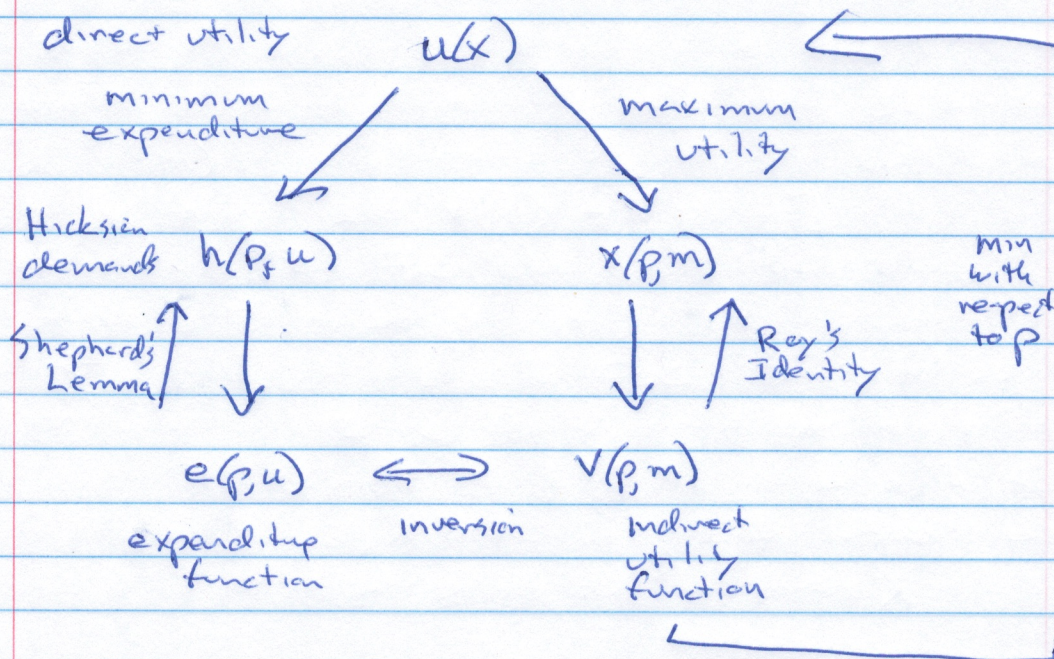
line through x' means that the consumer can get to a higher indifference curve than the one through x^* . Therefore $v(p, 1)$ will exceed $v(p^*, 1) = u(x^*)$.

When you are using this method to compute $u(x)$, what you will do in practice (assuming differentiability of $v(p, m)$) is set $m=1$, construct a Lagrangean of the form $L = v(p, 1) - d[px - 1]$ and look at the FOC when you differentiate with respect to the prices $p_1, \dots, p_n$. The constraint $\sum_i p_i x_i = 1$ is also a FOC. Don't forget that the x_i for $i=1, \dots, n$ are treated as constants! (you might be in the habit of differentiating with respect to x_i , but don't do that here).

If the algebra works out ~~just~~ nicely, you may be able to solve for the prices (and d) as functions of the x vector using the FOCs. Then substitute the resulting prices into $v(p, 1)$ which should give you a function that depends only on x (all the prices should be eliminated). This function is $u(x)$.

Keep in mind that this method assumes $u(x)$ is differentiable and quasi-concave (we assumed this when we used an inverse demand function). So if you are being super careful, you might check that your solution for $u(x)$ satisfies these conditions. However, this is usually not an issue.

Now that we know how to get $u(x)$ from $v(p, m)$, I want to give you a chart that shows how various concepts in consumer theory are related.



subject to
a utility
constraint

If you know the direct utility function, you can get the Hicksian demands $h(p, u)$ by minimizing expenditure and then substitute them into $p \cdot x$ to get the expenditure function $e(p, u)$. If you don't know $u(x)$ but you do know $e(p, u)$, you can get $h(p, u)$ from Shepherd's Lemma.

Similarly if you know $u(x)$, you can get the Marshallian demands $x(p, m)$ by maximizing utility subject to a budget constraint, and then substitute them into $u(x)$ to get the indirect utility function $v(p, m)$.

If you don't know $u(x)$ but you do know $v(p, m)$, you can get $x(p, m)$ from Roy's Identity.

It is easy to go back and forth between $e(p, u)$ and $v(p, m)$. If you know $e(p, u)$ set $m = e(p, u)$ and solve for u . This gives $u = v(p, m)$. If you know $v(p, m)$ set $u = v(p, m)$ and solve for m . This gives $m = e(p, u)$.

Finally if you know $v(p, m)$ but not $u(x)$, you can use the method described above to find $u(x)$.

That's all for Chapter 8!